Size polymorphism

Ensuring correction of array accesses with typing

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Requirements and context

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Safety critical embedded systems

- No errors at run-time
- Graphical specification (inference)
- Statically bounded memory

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Targeted array applications

- Signal processing, AI
- Non linear size relations, recursion
- Polymorphism (on size, on shape)

Arrays in programming languages

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Intensional arrays: indivisible objects

- Used with *iterators* (`map`, `fold`, ...)
- Correct accesses by construction (but limited expressiveness)
- Size inconsistencies (`zip`, `map2`, ...)

Agenda

1. Bringing sizes in types
    1. Refinements
    2. Size language
    3. Polymorphism
2. Size Inference
    1. Inference steps
    2. Size constraint resolution
3. The language
    1. Examples
    2. Coercions
    3. Binding time analysis

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Sizes in types

Size Polymorphism

$\texttt{val map} : \forall\iota.\ \forall\alpha,\beta.\ (\alpha \to \beta) \to \texttt{<}\iota\texttt{>} \to \texttt{[}\iota\texttt{]}\alpha \to \texttt{[}\iota\texttt{]}\beta$

Dependent Types

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1. Refinements

Sizes in types

❃ Refinements

General form [XP98, Fla06]

- Predicates over base type: $\{x : \mathtt{B} \mid P(x)\}$
- Sub-typing: predicate implication

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τ	$::=$		*Types*
	$\mid$	α, β, γ	variable
	$\mid$	`int`	integer
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Integer refinements

- $\texttt{<}\eta\texttt{>}$: singleton type $\{x : \texttt{int} \mid x = \eta\}$ (*size* η)
- $\texttt{[}\eta\texttt{]}$: interval type $\{x : \texttt{int} \mid 0 \leq x < \eta\}$ (*index* η)

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Arrays as functions[1] with bounded domain: $\texttt{[}\eta\texttt{]}\tau \equiv \texttt{[}\eta\texttt{]} \to \tau$

- Correctness of accesses ensured by typing

[1] for typing purposes only

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2. Size language

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Dependent Types

val map : $\forall\alpha,\beta.\ (\alpha \to \beta) \to \Pi n{:}\mathtt{int}.\ [n]\,\alpha \to [n]\,\beta$

1. Refinements
2. Expressions

Sizes in types

✻ Size language

Multivariate polynomials $\eta \in \mathbb{Z}[\mathcal{V}_\eta]$

η	$::=$		*Sizes*
	$\mid$	ι, δ, κ	variable
	$\mid$	n	constant
	$\mid$	$\eta + \eta$	sum
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Equivalent types schemes

$\texttt{val sample} : \forall \iota, \delta.\ \forall \alpha.\ \texttt{<}\delta\texttt{>} \rightarrow \texttt{[}\iota * \delta - \delta + 1\texttt{]}\alpha \rightarrow \texttt{[}\iota\texttt{]}\alpha$

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No most general types schemes

`let zero` : $\texttt{<}\iota\texttt{>} \to \texttt{<}0\texttt{>} = \lambda n : \texttt{<}\iota\texttt{>}.\ (n - 1) * (n - 2)$

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$\Longrightarrow$ Well-typed if $\iota = 1$ or $\iota = 2$

$$\text{Incompatible types} \begin{cases} \texttt{<}1\texttt{>} \to \texttt{<}0\texttt{>} \\ \texttt{<}2\texttt{>} \to \texttt{<}0\texttt{>} \end{cases}$$

Sizes in types

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1. Refinements
2. Expressions
3. Dependency

Sizes in types

✻ Polymorphism

σ	$::=$		*Type scheme*
	$\mid$	τ	simple type
	$\mid$	$\forall \iota .\ \sigma$	size quantif.
	$\mid$	$\forall \alpha .\ \sigma$	type quantif.

Handling sizes as types

- Static sizes only
- Constraint based definitions
- Implicit instantiation and generalization, inference

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Example: scalar product

$\texttt{val fold} : \forall\iota.\ \forall\alpha,\beta.\ (\alpha \to \beta \to \alpha) \to \texttt{<}\iota\texttt{>} \to \alpha \to \texttt{[}\iota\texttt{]}\beta \to \alpha$

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$\texttt{let dot\_product} : \_ = \lambda u : \_.\ \lambda v : \_.\ \texttt{fold}\ (+)\ \texttt{<\_>}\ 0\ (\texttt{map2}\ (*)\ \texttt{<\_>}\ u\ v)$

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$\texttt{val dot\_product} : \forall\iota.\ [\iota]\texttt{int} \to [\iota]\texttt{int} \to \texttt{int}$

Agenda

Inference

The `pack` operator

$$\texttt{val window} : \forall \iota, \kappa.\ \forall \alpha.\ \texttt{<}\kappa\texttt{>} \to \texttt{[}\iota + \kappa - 1\texttt{]}\,\alpha \to \texttt{[}\iota\texttt{]}\,\texttt{[}\kappa\texttt{]}\,\alpha$$
$$\texttt{val sample} : \forall \iota, \delta.\ \forall \alpha.\ \texttt{<}\delta\texttt{>} \to \texttt{[}\iota * \delta - \delta + 1\texttt{]}\,\alpha \to \texttt{[}\iota\texttt{]}\,\alpha$$

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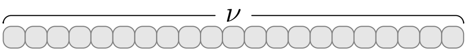
$\texttt{let pack} : \_ = \lambda x : \_ .$

$\quad \texttt{sample <\_> (window <\_>}\ x\texttt{)}$



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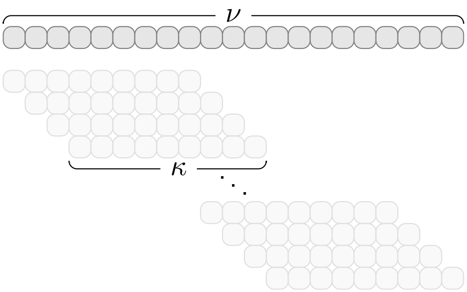

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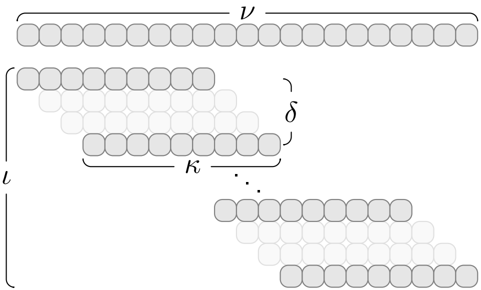

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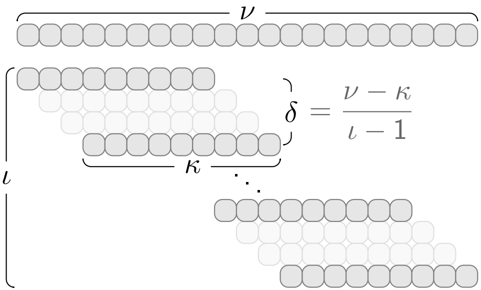

Inference

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▷ Type inference

▷ Integer refinement

▷ Size inference

Inference

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▷ **Type inference** ▷ Integer refinement ▷ Size inference

- Explicit instantiation

Inference

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$\texttt{let pack} : \forall \alpha.\ (\overline{\texttt{int}} \rightarrow \alpha) \rightarrow \overline{\texttt{int}} \rightarrow \overline{\texttt{int}} \rightarrow \alpha =$
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▷ **Type inference** ▷ Integer refinement ▷ Size inference

- Explicit instantiation
- Unrefined type $\overline{\texttt{int}}$
- Structural unification, generalization

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$\quad \lambda x : [\_]\alpha.$

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▷ Type inference

▷ **Integer refinement**

▷ Size inference

- $\overline{\texttt{int}}$ occurrence refinement
- Local propagation

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- ▷ Type inference
- ▷ Integer refinement
- ▷ **Size inference**

Inference

Instantiating `pack`

$$\texttt{val window} : \forall \iota, \kappa.\ \forall \alpha.\ \texttt{<}\kappa\texttt{>} \rightarrow [\iota + \kappa - 1]\alpha \rightarrow [\iota][\kappa]\alpha$$
$$\texttt{val sample} : \forall \iota, \delta.\ \forall \alpha.\ \texttt{<}\delta\texttt{>} \rightarrow [\iota * \delta - \delta + 1]\alpha \rightarrow [\iota]\alpha$$

$$\delta = \frac{\nu - \kappa}{\iota - 1}$$

$$\texttt{let pack} : \forall \alpha.\ [\nu]\alpha \rightarrow [\iota][\kappa]\alpha =$$
$$\quad \lambda x : [\nu_i]\alpha.$$
$$\quad \texttt{sample}_{\ \iota_s\ \delta_s\ [\kappa'_w]\alpha}\ \texttt{<}\kappa_1\texttt{>}$$
$$\quad\quad (\texttt{window}_{\ \iota_w\ \kappa_w\ \alpha}\ \texttt{<}\kappa_2\texttt{>}\ x)$$

▷ Type inference ▷ Integer refinement ▷ **Size inference**

- Explicit instantiation
- Collect constraints of the form $\eta = 0$

Inference

Instantiating `pack`

$\texttt{val window} : \forall\iota,\kappa.\ \forall\alpha.\ \texttt{<}\kappa\texttt{>} \rightarrow [\iota+\kappa-1]\alpha \rightarrow [\iota][\kappa]\alpha$

$\texttt{val sample} : \forall\iota,\delta.\ \forall\alpha.\ \texttt{<}\delta\texttt{>} \rightarrow [\iota*\delta-\delta+1]\alpha \rightarrow [\iota]\alpha$

$$\delta = \frac{\nu-\kappa}{\iota-1}$$

$\texttt{let pack} : \forall\iota,\kappa,\delta.\ \forall\alpha.\ [\iota*\delta-\delta+\kappa]\alpha \rightarrow [\iota][\kappa]\alpha =$

$\quad \lambda x : [\iota*\delta-\delta+\kappa]\alpha.$

$\quad \texttt{sample}_{\,\iota\ \delta\ [\kappa]\alpha}\ \texttt{<}\delta\texttt{>}$

$\qquad (\texttt{window}_{\,(\iota*\delta-\delta+1)\ \kappa\ \alpha}\ \texttt{<}\kappa\texttt{>}\ x)$

▷ Type inference ▷ Integer refinement ▷ **Size inference**

- Explicit instantiation
- Collect constraints of the form $\eta = 0$
- Resolve system at generalization points
- Isolated variable elimination: $\iota - \eta = 0,\ \iota \notin \mathit{Vars}(\eta)$

Inference

Instantiating `pack`

$$\texttt{val window} : \forall \iota, \kappa.\ \forall \alpha.\ \texttt{<}\kappa\texttt{>} \to [\iota + \kappa - 1]\alpha \to [\iota]\,[\kappa]\alpha$$
$$\texttt{val sample} : \forall \iota, \delta.\ \forall \alpha.\ \texttt{<}\delta\texttt{>} \to [\iota * \delta - \delta + 1]\alpha \to [\iota]\alpha$$

$$\delta = \frac{\nu - \kappa}{\iota - 1}$$

$$\texttt{val pack} : \forall \iota, \kappa, \delta.\ \forall \alpha.\ [\iota * \delta - \delta + \kappa]\alpha \to [\iota]\,[\kappa]\alpha$$

Inference

Instantiating `pack`

$$\texttt{val window} : \forall \iota, \kappa.\ \forall \alpha.\ \texttt{<}\kappa\texttt{>} \rightarrow [\iota + \kappa - 1]\alpha \rightarrow [\iota][\kappa]\alpha$$

$$\texttt{val sample} : \forall \iota, \delta.\ \forall \alpha.\ \texttt{<}\delta\texttt{>} \rightarrow [\iota * \delta - \delta + 1]\alpha \rightarrow [\iota]\alpha$$

$$\texttt{val pack} : \forall \iota, \kappa, \delta.\ \forall \alpha.\ [\iota * \delta - \delta + \kappa]\alpha \rightarrow [\iota][\kappa]\alpha$$

$$\texttt{let split} : [\iota * \kappa]\_ \rightarrow [\iota][\kappa]\_ = \texttt{pack}$$

Inference

Instantiating `pack`

$$\texttt{val window} : \forall\iota,\kappa.\ \forall\alpha.\ \texttt{<}\kappa\texttt{>} \to [\iota+\kappa-1]\alpha \to [\iota][\kappa]\alpha$$

$$\texttt{val sample} : \forall\iota,\delta.\ \forall\alpha.\ \texttt{<}\delta\texttt{>} \to [\iota*\delta-\delta+1]\alpha \to [\iota]\alpha$$

$$\texttt{val pack} : \forall\iota,\kappa,\delta.\ \forall\alpha.\ [\iota*\delta-\delta+\kappa]\alpha \to [\iota][\kappa]\alpha$$

$$\texttt{let split} : \forall\alpha.\ [\iota*\kappa]\alpha \to [\iota][\kappa]\alpha = \texttt{pack}_{\ \iota'\ \kappa'\ \delta\ \alpha}$$

▷ **Size inference**

Inference

Instantiating `pack`

$$\texttt{val window} : \forall \iota, \kappa.\ \forall \alpha.\ \texttt{<}\kappa\texttt{>} \to [\iota + \kappa - 1]\alpha \to [\iota]\,[\kappa]\alpha$$
$$\texttt{val sample} : \forall \iota, \delta.\ \forall \alpha.\ \texttt{<}\delta\texttt{>} \to [\iota * \delta - \delta + 1]\alpha \to [\iota]\alpha$$

$$\texttt{val pack} : \forall \iota, \kappa, \delta.\ \forall \alpha.\ [\iota * \delta - \delta + \kappa]\alpha \to [\iota]\,[\kappa]\alpha$$

$$\texttt{let split} : \forall \alpha.\ [\iota * \kappa]\alpha \to [\iota]\,[\kappa]\alpha = \texttt{pack}_{\iota\ \kappa\ \delta\ \alpha}$$

▷ **Size inference**

1. Variable elimination (equivalent substitution)

$$\iota * \delta - \delta + \kappa = \iota * \kappa$$

Inference

Instantiating `pack`

$$\texttt{val window} : \forall\iota,\kappa.\ \forall\alpha.\ \texttt{<}\kappa\texttt{>} \to [\iota+\kappa-1]\,\alpha \to [\iota]\,[\kappa]\,\alpha$$
$$\texttt{val sample} : \forall\iota,\delta.\ \forall\alpha.\ \texttt{<}\delta\texttt{>} \to [\iota*\delta-\delta+1]\,\alpha \to [\iota]\,\alpha$$

$$\texttt{val pack} : \forall\iota,\kappa,\delta.\ \forall\alpha.\ [\iota*\delta-\delta+\kappa]\,\alpha \to [\iota]\,[\kappa]\,\alpha$$

$$\texttt{let split} : \forall\alpha.\ [\iota*\kappa]\,\alpha \to [\iota]\,[\kappa]\,\alpha = \texttt{pack}_{\iota\ \kappa\ \delta\ \alpha}$$

- **Size inference**
  1. Variable elimination (equivalent substitution)

$$\iota*\delta-\delta+\kappa = \iota*\kappa$$

  2. Structural unification (nonequivalent substitution)

$$(\iota-1)*(\delta-\kappa) = 0$$

Inference

Instantiating `pack`

$$\texttt{val window} : \forall\iota,\kappa.\ \forall\alpha.\ \texttt{<}\kappa\texttt{>} \to [\iota+\kappa-1]\alpha \to [\iota][\kappa]\alpha$$
$$\texttt{val sample} : \forall\iota,\delta.\ \forall\alpha.\ \texttt{<}\delta\texttt{>} \to [\iota*\delta-\delta+1]\alpha \to [\iota]\alpha$$

$$\texttt{val pack} : \forall\iota,\kappa,\delta.\ \forall\alpha.\ [\iota*\delta-\delta+\kappa]\alpha \to [\iota][\kappa]\alpha$$

$$\texttt{let split} : \forall\iota,\kappa.\ \forall\alpha.\ [\iota*\kappa]\alpha \to [\iota][\kappa]\alpha = \texttt{pack}_{\ \iota\ \kappa\ \kappa\ \alpha}$$

▷ **Size inference**

1. Variable elimination (equivalent substitution)

$$\iota*\delta-\delta+\kappa = \iota*\kappa$$

2. Structural unification (nonequivalent substitution)

$$(\iota-1)*(\delta-\kappa) = 0$$

Agenda

1. Bringing sizes in types
    1. Refinements
    2. Size language
    3. Polymorphism

2. Size Inference
    1. Inference steps
    2. Size constraint resolution

3. The language
    1. Examples
    2. Coercions
    3. Binding time analysis

Expressions (I)

Extensional array use [SSSV17]

e	$::=$		*Expressions*
	$\mid$	x	variable
	$\mid$	$e\ e$	application
	$\mid$	$\lambda x : \tau.\ e$	abstraction
	$\mid$	$\texttt{true} \mid \texttt{false}$	boolean
	$\mid$	n	integer
	$\mid$	$<\eta>$	size

Expressions (I)

Extensional array use [SSSV17]

$\texttt{val map} : \forall \iota.\ \forall \alpha, \beta.\ (\alpha \to \beta) \to \texttt{<}\iota\texttt{>} \to [\iota]\alpha \to [\iota]\beta$

e	$::=$		*Expressions*
	$\mid$	x	variable
	$\mid$	$e\ e$	application
	$\mid$	$\lambda x : \tau.\ e$	abstraction
	$\mid$	$\texttt{true} \mid \texttt{false}$	boolean
	$\mid$	n	integer
	$\mid$	$\texttt{<}\eta\texttt{>}$	size

Expressions (I)

Extensional array use [SSSV17]

val map : $\forall\iota.\ \forall\alpha,\beta.\ (\alpha \to \beta) \to \texttt{<}\iota\texttt{>} \to \texttt{[}\iota\texttt{]}\alpha \to \texttt{[}\iota\texttt{]}\beta$

let map : $\_ = \lambda f : \_.\ \lambda n : \texttt{<}\iota\texttt{>}.\ \lambda X : \_.\ \lambda i : \texttt{[}\iota\texttt{]}.\ f\ (X\ i)$

e	$::=$		*Expressions*
	$\mid$	x	variable
	$\mid$	$e\ e$	application
	$\mid$	$\lambda x : \tau.\ e$	abstraction
	$\mid$	true $\mid$ false	boolean
	$\mid$	n	integer
	$\mid$	$\texttt{<}\eta\texttt{>}$	size

Expressions (I)

Extensional array use [SSSV17]

$\texttt{val map} : \forall\iota.\ \forall\alpha,\beta.\ (\alpha \to \beta) \to \texttt{<}\iota\texttt{>} \to [\iota]\alpha \to [\iota]\beta$

$\texttt{let map} : \_ = \lambda f : \_.\ \lambda n : \texttt{<}\iota\texttt{>}.\ \lambda X : \_.\ \lambda i : [\iota].\ f\ (X\ i)$

$\texttt{val window} : \forall\iota,\kappa.\ \forall\alpha.\ \texttt{<}\kappa\texttt{>} \to [\iota + \kappa - 1]\alpha \to [\iota][\kappa]\alpha$

e	$::=$		*Expressions*
	$\mid$	x	variable
	$\mid$	$e\ e$	application
	$\mid$	$\lambda x : \tau.\ e$	abstraction
	$\mid$	$\texttt{true} \mid \texttt{false}$	boolean
	$\mid$	n	integer
	$\mid$	$\texttt{<}\eta\texttt{>}$	size

Expressions (I)

Extensional array use [SSSV17]

$\texttt{val map} : \forall\iota.\ \forall\alpha,\beta.\ (\alpha \to \beta) \to \texttt{<}\iota\texttt{>} \to \texttt{[}\iota\texttt{]}\alpha \to \texttt{[}\iota\texttt{]}\beta$

$\texttt{let map} : \_ = \lambda f : \_.\ \lambda n : \texttt{<}\iota\texttt{>}.\ \lambda X : \_.\ \lambda i : \texttt{[}\iota\texttt{]}.\ f\ (X\ i)$

$\texttt{val window} : \forall\iota,\kappa.\ \forall\alpha.\ \texttt{<}\kappa\texttt{>} \to \texttt{[}\iota + \kappa - 1\texttt{]}\alpha \to \texttt{[}\iota\texttt{]}\,\texttt{[}\kappa\texttt{]}\alpha$

$\texttt{let window} : \_ = \lambda k : \texttt{<}\kappa\texttt{>}.\ \lambda X : \texttt{[}\iota + \kappa - 1\texttt{]}\_.\ \lambda i : \texttt{[}\iota\texttt{]}.\ \lambda j : \texttt{[}\kappa\texttt{]}.\ X\ (i + j \triangleright \texttt{[}\_\texttt{]})$

e	$::=$		*Expressions*
	$\mid$	x	variable
	$\mid$	$e\ e$	application
	$\mid$	$\lambda x : \tau.\ e$	abstraction
	$\mid$	$\texttt{true} \mid \texttt{false}$	boolean
	$\mid$	n	integer
	$\mid$	$\texttt{<}\eta\texttt{>}$	size

Expressions (I)

Extensional array use [SSSV17]

$\texttt{val map} : \forall\iota.\ \forall\alpha,\beta.\ (\alpha \to \beta) \to \texttt{<}\iota\texttt{>} \to [\iota]\alpha \to [\iota]\beta$

$\texttt{let map} : \_ = \lambda f : \_.\ \lambda n : \texttt{<}\iota\texttt{>}.\ \lambda X : \_.\ \lambda i : [\iota].\ f\ (X\ i)$

$\texttt{val window} : \forall\iota,\kappa.\ \forall\alpha.\ \texttt{<}\kappa\texttt{>} \to [\iota + \kappa - 1]\alpha \to [\iota]\,[\kappa]\alpha$

$\texttt{let window} : \_ = \lambda k : \texttt{<}\kappa\texttt{>}.\ \lambda X : [\iota + \kappa - 1]\_.\ \lambda i : [\iota].\ \lambda j : [\kappa].\ X\ (i + j \triangleright [\_])$

Coercions: post-typing checks [Fla06, HE21]

e	$::=$		*Expressions*
	$\mid$	x	variable
	$\mid$	$e\ e$	application
	$\mid$	$\lambda x : \tau.\ e$	abstraction
	$\mid$	$\texttt{true} \mid \texttt{false}$	boolean
	$\mid$	n	integer
	$\mid$	$\texttt{<}\eta\texttt{>}$	size
	$\mid$	$e \triangleright \tau$	coercion

Expressions (I)

Extensional array use [SSSV17]

$\texttt{val map} : \forall\iota.\ \forall\alpha,\beta.\ (\alpha \to \beta) \to \texttt{<}\iota\texttt{>} \to \texttt{[}\iota\texttt{]}\alpha \to \texttt{[}\iota\texttt{]}\beta$

$\texttt{let map} : \_ = \lambda f : \_.\ \lambda n : \texttt{<}\iota\texttt{>}.\ \lambda X : \_.\ \lambda i : \texttt{[}\iota\texttt{]}.\ f\ (X\ i)$

$\texttt{val window} : \forall\iota,\kappa.\ \forall\alpha.\ \texttt{<}\kappa\texttt{>} \to \texttt{[}\iota + \kappa - 1\texttt{]}\alpha \to \texttt{[}\iota\texttt{]}\,\texttt{[}\kappa\texttt{]}\alpha$

$\texttt{let window} : \_ = \lambda k : \texttt{<}\kappa\texttt{>}.\ \lambda X : \texttt{[}\iota + \kappa - 1\texttt{]}\_.\ \lambda i : \texttt{[}\iota\texttt{]}.\ \lambda j : \texttt{[}\kappa\texttt{]}.\ X\ (i + j \triangleright \texttt{[}\_\texttt{]})$

Coercions: post-typing checks [Fla06, HE21]

- Integer upcasting

$$(e : \texttt{int}) \triangleright \texttt{<}\eta\texttt{>}$$
$$(e : \texttt{int}) \triangleright \texttt{[}\eta\texttt{]}$$

e	$::=$		*Expressions*
	$\mid$	x	variable
	$\mid$	$e\ e$	application
	$\mid$	$\lambda x : \tau.\ e$	abstraction
	$\mid$	$\texttt{true} \mid \texttt{false}$	boolean
	$\mid$	n	integer
	$\mid$	$\texttt{<}\eta\texttt{>}$	size
	$\mid$	$e \triangleright \tau$	coercion

Expressions (I)

Extensional array use [SSSV17]

$\texttt{val map} : \forall\iota.\ \forall\alpha,\beta.\ (\alpha \to \beta) \to \texttt{<}\iota\texttt{>} \to \texttt{[}\iota\texttt{]}\alpha \to \texttt{[}\iota\texttt{]}\beta$

$\texttt{let map} : \_ = \lambda f:\_.\ \lambda n:\texttt{<}\iota\texttt{>}.\ \lambda X:\_.\ \lambda i:\texttt{[}\iota\texttt{]}.\ f\ (X\ i)$

$\texttt{val window} : \forall\iota,\kappa.\ \forall\alpha.\ \texttt{<}\kappa\texttt{>} \to \texttt{[}\iota+\kappa-1\texttt{]}\alpha \to \texttt{[}\iota\texttt{]}\,\texttt{[}\kappa\texttt{]}\alpha$

$\texttt{let window} : \_ = \lambda k:\texttt{<}\kappa\texttt{>}.\ \lambda X:\texttt{[}\iota+\kappa-1\texttt{]}\_.\ \lambda i:\texttt{[}\iota\texttt{]}.\ \lambda j:\texttt{[}\kappa\texttt{]}.\ X\ (i+j \triangleright \texttt{[}\_\texttt{]})$

Coercions: post-typing checks [Fla06, HE21]

- Integer upcasting

$$(e : \texttt{int}) \triangleright \texttt{<}\eta\texttt{>}$$
$$(e : \texttt{int}) \triangleright \texttt{[}\eta\texttt{]}$$

- Size conversion

$$(e : \tau) \triangleright \tau', \text{ if } \tau \approx \tau'$$

$\approx$: Size ignoring comparison

e	$::=$		*Expressions*
	$\mid$	x	variable
	$\mid$	$e\ e$	application
	$\mid$	$\lambda x:\tau.\ e$	abstraction
	$\mid$	$\texttt{true} \mid \texttt{false}$	boolean
	$\mid$	n	integer
	$\mid$	$\texttt{<}\eta\texttt{>}$	size
	$\mid$	$e \triangleright \tau$	coercion

Expressions (II)

Fast Fourier Transform

e	$::=$		*Expressions*
	$\mid$	x	variable
	$\mid$	$e\ e$	application
	$\mid$	$\lambda x : \tau.\ e$	abstraction
	$\mid$	$\texttt{true} \mid \texttt{false}$	boolean
	$\mid$	n	integer
	$\mid$	$<\eta>$	size
	$\mid$	$e \triangleright \tau$	coercion

Expressions (II)

Fast Fourier Transform

$\texttt{val gft} : \forall\iota.\ [\iota]\,\texttt{cpx} \to [\iota]\,\texttt{cpx}$ $\qquad \mathcal{O}(\iota) = \iota^2$

$\texttt{val fft} : \forall\iota.\ ([\iota]\,\texttt{cpx} \to [\iota]\,\texttt{cpx}) \to [2\iota]\,\texttt{cpx} \to [2\iota]\,\texttt{cpx}$ $\qquad \mathcal{O}(\iota) = 2\mathcal{O}_f + 2\iota$

e	::=		*Expressions*
	|	x	variable
	|	$e\ e$	application
	|	$\lambda x : \tau .\ e$	abstraction
	|	`true` | `false`	boolean
	|	n	integer
	|	$\langle \eta \rangle$	size
	|	$e \triangleright \tau$	coercion

Expressions (II)

Fast Fourier Transform

$\texttt{val gft} : \forall\iota.\ [\iota]\,\texttt{cpx} \to [\iota]\,\texttt{cpx}$ $\qquad \mathcal{O}(\iota) = \iota^2$

$\texttt{val fft} : \forall\iota.\ ([\iota]\,\texttt{cpx} \to [\iota]\,\texttt{cpx}) \to [2\iota]\,\texttt{cpx} \to [2\iota]\,\texttt{cpx}$ $\qquad \mathcal{O}(\iota) = 2\mathcal{O}_f + 2\iota$

$\texttt{let dft} : \underline{\ \ } = \ldots$

e	$::=$		*Expressions*
	$\mid$	x	variable
	$\mid$	$e\ e$	application
	$\mid$	$\lambda x : \tau.\ e$	abstraction
	$\mid$	$\texttt{true} \mid \texttt{false}$	boolean
	$\mid$	n	integer
	$\mid$	$<\eta>$	size
	$\mid$	$e \triangleright \tau$	coercion

Expressions (II)

Fast Fourier Transform

$\texttt{val gft} : \forall\iota.\ [\iota]\,\texttt{cpx} \to [\iota]\,\texttt{cpx}$ $\qquad \mathcal{O}(\iota) = \iota^2$

$\texttt{val fft} : \forall\iota.\ ([\iota]\,\texttt{cpx} \to [\iota]\,\texttt{cpx}) \to [2\iota]\,\texttt{cpx} \to [2\iota]\,\texttt{cpx}$ $\qquad \mathcal{O}(\iota) = 2\mathcal{O}_f + 2\iota$

$\texttt{let dft} : \_ = \texttt{fix}\ f : \forall\iota.\ [\iota]\,\_ \to [\iota]\,\_ = \lambda X : \_.$

$\quad \ldots$

Polymorphic recursion [Mee83, Myc84]

e	$::=$		*Expressions*
	$\mid$	x	variable
	$\mid$	$e\ e$	application
	$\mid$	$\lambda x : \tau.\ e$	abstraction
	$\mid$	$\texttt{true} \mid \texttt{false}$	boolean
	$\mid$	n	integer
	$\mid$	$<\eta>$	size
	$\mid$	$e \triangleright \tau$	coercion
	$\mid$	$\texttt{fix}\ x : \sigma = e$	fix-point

Expressions (II)

Fast Fourier Transform

$\texttt{val gft} : \forall\iota.\ [\iota]\,\texttt{cpx} \to [\iota]\,\texttt{cpx}$ $\qquad \mathcal{O}(\iota) = \iota^2$

$\texttt{val fft} : \forall\iota.\ ([\iota]\,\texttt{cpx} \to [\iota]\,\texttt{cpx}) \to [2\iota]\,\texttt{cpx} \to [2\iota]\,\texttt{cpx}$ $\qquad \mathcal{O}(\iota) = 2\mathcal{O}_f + 2\iota$

$\texttt{let dft} : \_ = \texttt{fix}\ f : \forall\iota.\ [\iota]\,\_ \to [\iota]\,\_ = \lambda X : \_.$

$\quad \ldots$

Polymorphic recursion [Mee83, Myc84]

e	$::=$		*Expressions*
	$\mid$	x	variable
	$\mid$	$e\ e$	application
	$\mid$	$\lambda x : \tau.\ e$	abstraction
	$\mid$	$\texttt{true} \mid \texttt{false}$	boolean
	$\mid$	n	integer
	$\mid$	$<\eta>$	size
	$\mid$	$e \triangleright \tau$	coercion
	$\mid$	$\texttt{fix}\ x : \sigma = e$	fix-point
	$\mid$	$\texttt{let}\ x : \sigma = e\ \texttt{in}\ e$	local def

Fast Fourier Transform

val gft : $\forall\iota.\ [\iota]\,\mathrm{cpx} \to [\iota]\,\mathrm{cpx}$ $\mathcal{O}(\iota) = \iota^2$

val fft : $\forall\iota.\ ([\iota]\,\mathrm{cpx} \to [\iota]\,\mathrm{cpx}) \to [2\iota]\,\mathrm{cpx} \to [2\iota]\,\mathrm{cpx}$ $\mathcal{O}(\iota) = 2\mathcal{O}_f + 2\iota$

```
let dft : _ = fix f:∀ι. [ι]_ → [ι]_ = λX:_.
    let size ν = <ι>/2 in
    ...
```

Polymorphic recursion [Mee83, Myc84]

Local size existential quantification

e	::=		*Expressions*
	|	x	variable
	|	$e\ e$	application
	|	$\lambda x : \tau.\ e$	abstraction
	|	true | false	boolean
	|	n	integer
	|	$<\eta>$	size
	|	$e \triangleright \tau$	coercion
	|	fix $x : \sigma = e$	fix-point
	|	let $x : \sigma = e$ in e	local def
	|	let size $\iota = e$ in e	size def

Expressions (II)

Fast Fourier Transform

```
val gft : ∀ι. [ι] cpx → [ι] cpx                                  𝒪(ι) = ι²
val fft : ∀ι. ([ι] cpx → [ι] cpx) → [2ι] cpx → [2ι] cpx          𝒪(ι) = 2𝒪_f + 2ι
```

```
let dft : _ = fix f : ∀ι. [ι]_ → [ι]_ = λX : _.
    let size ν = <ι>/2 in
    case <ι> ≠ <2ν>
    then ...
    else ...
```

Polymorphic recursion [Mee83, Myc84]

Local size existential quantification

Statically resolved branching

e ::=		*Expressions*
		x	variable
		$e\ e$	application
		$\lambda x : \tau.\ e$	abstraction
		`true` | `false`	boolean
		n	integer
		$<\eta>$	size
		$e \triangleright \tau$	coercion
		`fix` $x : \sigma = e$	fix-point
		`let` $x : \sigma = e$ `in` e	local def
		`let size` $\iota = e$ `in` e	size def
		`case` e `then` e `else` e	cases

Expressions (II)

Fast Fourier Transform

$\texttt{val gft} : \forall\iota.\ [\iota]\,\texttt{cpx} \to [\iota]\,\texttt{cpx}$ $\qquad \mathcal{O}(\iota) = \iota^2$

$\texttt{val fft} : \forall\iota.\ ([\iota]\,\texttt{cpx} \to [\iota]\,\texttt{cpx}) \to [2\iota]\,\texttt{cpx} \to [2\iota]\,\texttt{cpx}$ $\qquad \mathcal{O}(\iota) = 2\mathcal{O}_f + 2\iota$

$\texttt{let dft} : \_ = \texttt{fix}\ f : \forall\iota.\ [\iota]\,\_ \to [\iota]\,\_ = \lambda X : \_.$
$\quad \texttt{let size}\ \nu = \texttt{<}\iota\texttt{>}/2\ \texttt{in}$
$\quad \texttt{case}\ \texttt{<}\iota\texttt{>} \neq \texttt{<}2\nu\texttt{>}$
$\quad \texttt{then gft}\ X$
$\quad \texttt{else}\ \ldots$

Polymorphic recursion [Mee83, Myc84]

Local size existential quantification

Statically resolved branching

e	::=		*Expressions*
	|	x	variable
	|	$e\ e$	application
	|	$\lambda x : \tau.\ e$	abstraction
	|	true | false	boolean
	|	n	integer
	|	$\texttt{<}\eta\texttt{>}$	size
	|	$e \triangleright \tau$	coercion
	|	$\texttt{fix}\ x : \sigma = e$	fix-point
	|	$\texttt{let}\ x : \sigma = e\ \texttt{in}\ e$	local def
	|	$\texttt{let size}\ \iota = e\ \texttt{in}\ e$	size def
	|	$\texttt{case}\ e\ \texttt{then}\ e\ \texttt{else}\ e$	cases

Expressions (II)

Fast Fourier Transform

$\texttt{val gft} : \forall\iota.\ [\iota]\,\texttt{cpx} \to [\iota]\,\texttt{cpx}$ $\qquad \mathcal{O}(\iota) = \iota^2$

$\texttt{val fft} : \forall\iota.\ ([\iota]\,\texttt{cpx} \to [\iota]\,\texttt{cpx}) \to [2\iota]\,\texttt{cpx} \to [2\iota]\,\texttt{cpx}$ $\qquad \mathcal{O}(\iota) = 2\mathcal{O}_f + 2\iota$

$$\begin{array}{l}
\texttt{let dft} : _ = \texttt{fix}\ f : \forall\iota.\ [\iota]_ \to [\iota]_ = \lambda X : _. \\
\quad \texttt{let size}\ \nu = \texttt{<}\iota\texttt{>}/2\ \texttt{in} \\
\quad \texttt{case}\ \texttt{<}\iota\texttt{>} \neq \texttt{<}2\nu\texttt{>} \\
\quad \texttt{then gft}\ X \\
\quad \texttt{else fft}\ f\ X
\end{array}$$

e	$::=$		*Expressions*
	$\mid$	x	variable
	$\mid$	$e\ e$	application
	$\mid$	$\lambda x : \tau.\ e$	abstraction
	$\mid$	$\texttt{true} \mid \texttt{false}$	boolean
	$\mid$	n	integer
	$\mid$	$\texttt{<}\eta\texttt{>}$	size
	$\mid$	$e \triangleright \tau$	coercion
	$\mid$	$\texttt{fix}\ x : \sigma = e$	fix-point
	$\mid$	$\texttt{let}\ x : \sigma = e\ \texttt{in}\ e$	local def
	$\mid$	$\texttt{let size}\ \iota = e\ \texttt{in}\ e$	size def
	$\mid$	$\texttt{case}\ e\ \texttt{then}\ e\ \texttt{else}\ e$	cases

Polymorphic recursion [Mee83, Myc84]

Local size existential quantification

Statically resolved branching

Expressions (II)

Fast Fourier Transform

val gft : $\forall\iota.\ [\iota]\,$cpx $\to [\iota]\,$cpx $\qquad \mathcal{O}(\iota) = \iota^2$

val fft : $\forall\iota.\ ([\iota]\,$cpx $\to [\iota]\,$cpx$) \to [2\iota]\,$cpx $\to [2\iota]\,$cpx $\qquad \mathcal{O}(\iota) = 2\mathcal{O}_f + 2\iota$

```
let dft : _ = fix f:∀ι. [ι]_ → [ι]_ = λX:_.
    let size ν = <ι>/2 in
    case <ι> ≠ <2ν>
    then gft X
    else fft f (X ▷ [2ν]_)
```

Polymorphic recursion [Mee83, Myc84]

Local size existential quantification

Statically resolved branching

e	::=		*Expressions*
	|	x	variable
	|	$e\ e$	application
	|	$\lambda x : \tau.\ e$	abstraction
	|	true | false	boolean
	|	n	integer
	|	$<\eta>$	size
	|	$e \triangleright \tau$	coercion
	|	fix $x : \sigma = e$	fix-point
	|	let $x : \sigma = e$ in e	local def
	|	let size $\iota = e$ in e	size def
	|	case e then e else e	cases

Expressions (II)

Fast Fourier Transform

$\texttt{val gft} : \forall\iota.\ [\iota]\,\texttt{cpx} \to [\iota]\,\texttt{cpx}$ $\qquad \mathcal{O}(\iota) = \iota^2$

$\texttt{val fft} : \forall\iota.\ ([\iota]\,\texttt{cpx} \to [\iota]\,\texttt{cpx}) \to [2\iota]\,\texttt{cpx} \to [2\iota]\,\texttt{cpx}$ $\qquad \mathcal{O}(\iota) = 2\mathcal{O}_f + 2\iota$

$\texttt{let dft} : \_ = \texttt{fix}\ f : \forall\iota.\ [\iota]\_ \to [\iota]\_ = \lambda X : \_.$
$\quad \texttt{let size}\ \nu = \texttt{<}\iota\texttt{>}/2\ \texttt{in}$
$\quad \texttt{case}\ \texttt{<}\iota\texttt{>} \neq \texttt{<}2\nu\texttt{>}$
$\quad \texttt{then gft}\ X$
$\quad \texttt{else fft}\ f\ (X \triangleright [2\nu]\_) \triangleright [\iota]\_$

Polymorphic recursion [Mee83, Myc84]

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	$\mid$	$\texttt{fix}\ x : \sigma = e$	fix-point
	$\mid$	$\texttt{let}\ x : \sigma = e\ \texttt{in}\ e$	local def
	$\mid$	$\texttt{let size}\ \iota = e\ \texttt{in}\ e$	size def
	$\mid$	$\texttt{case}\ e\ \texttt{then}\ e\ \texttt{else}\ e$	cases

Coercions

Purposes

- Separating array accesses from property checking (bounds, ...)
- Bypass size language expressiveness limitations
- Simplify typing (avoid *ad-hoc* rules)
- While rarely needed (if some primitives are available: `window`, `sample`, ...)

Coercions

Purposes

- Separating array accesses from property checking (bounds, ...)
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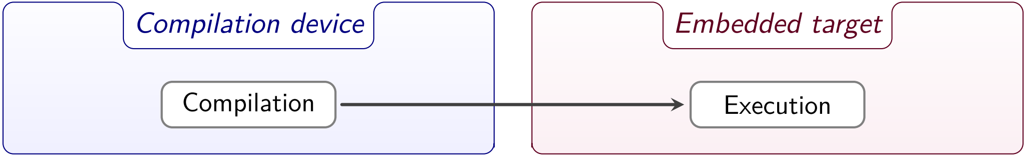
- While rarely needed (if some primitives are available: `window`, `sample`, ...)

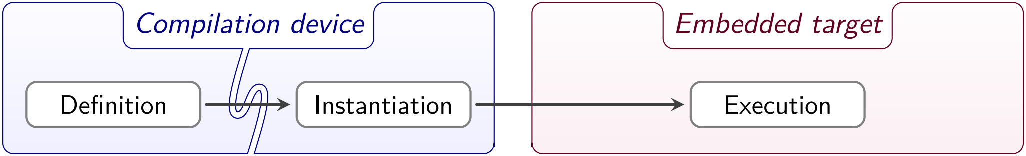
Handling coercion errors

- Defensive code generation
- Advanced formal analysis (abstract interpretation, SMT solvers)
- Binding time restriction (static sizes)

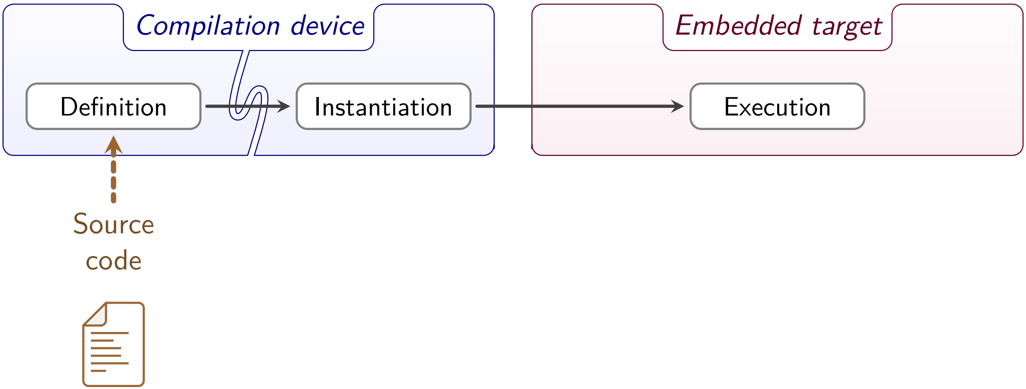
Embedded Synchronous Program Timeline



Embedded Synchronous Program Timeline



Embedded Synchronous Program Timeline



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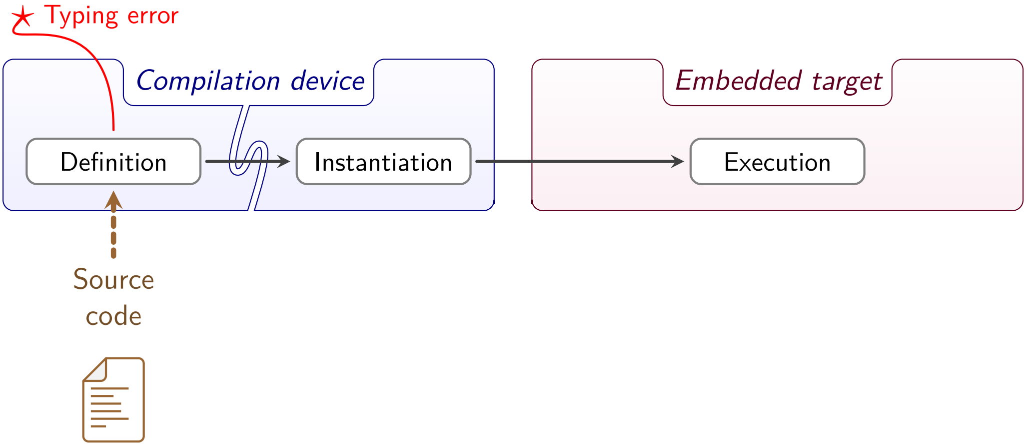

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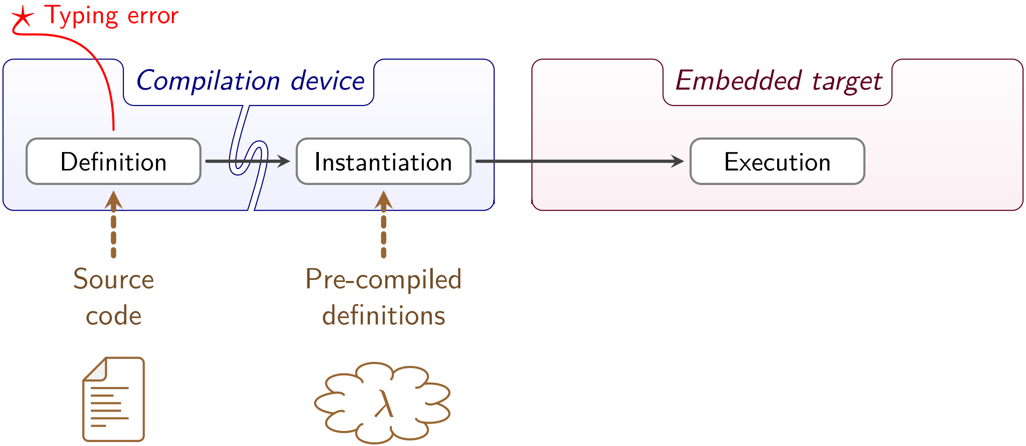
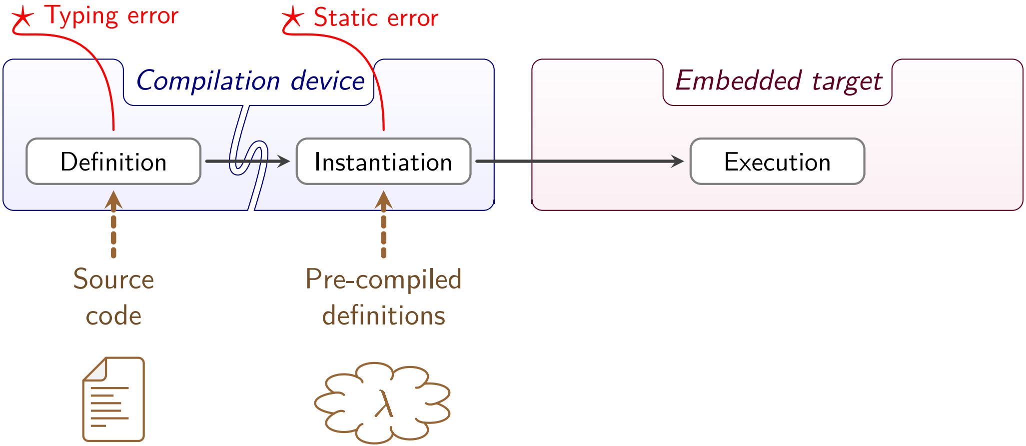
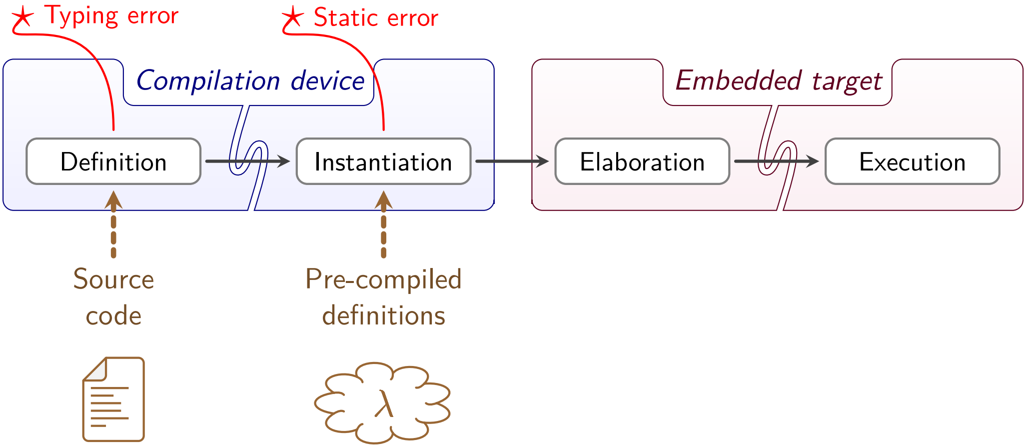


Embedded Synchronous Program Timeline



Embedded Synchronous Program Timeline



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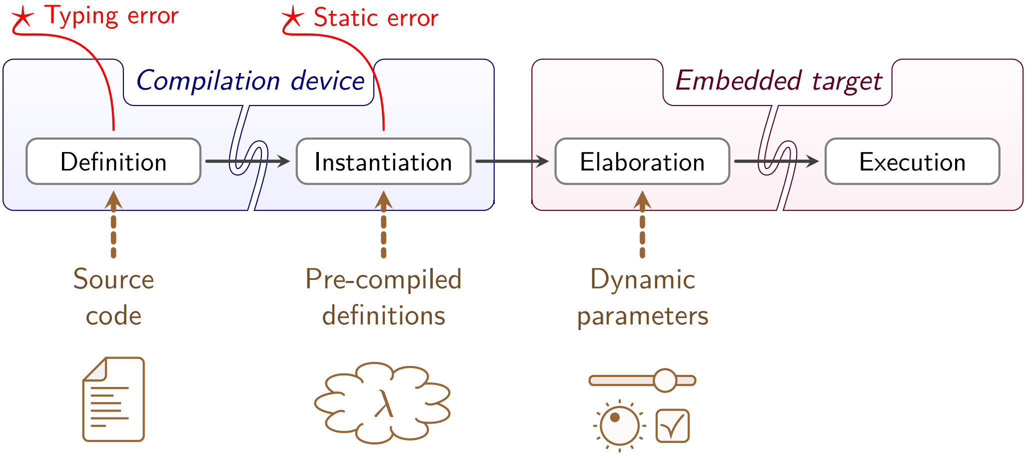
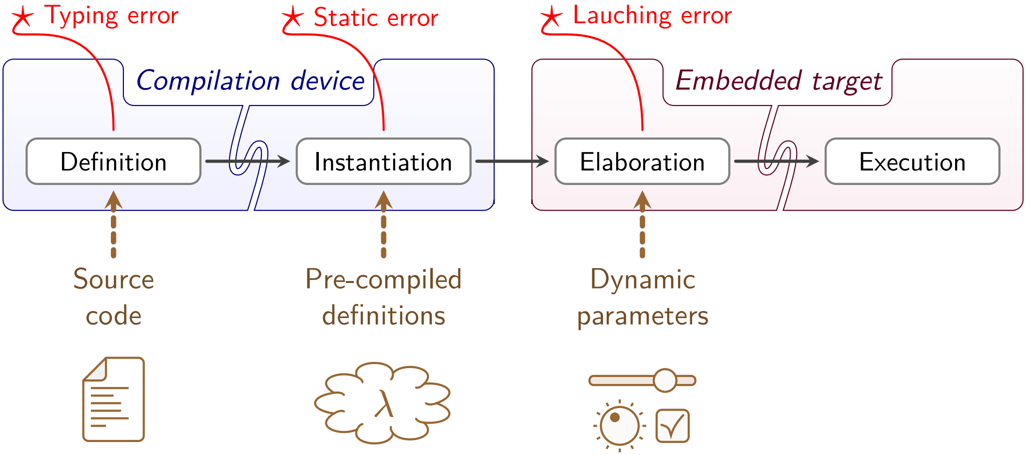
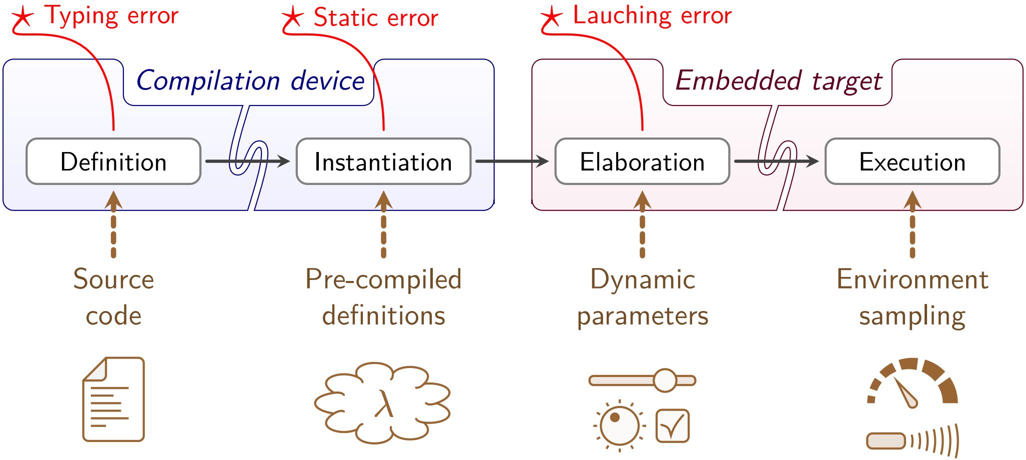


Embedded Synchronous Program Timeline



Embedded Synchronous Program Timeline



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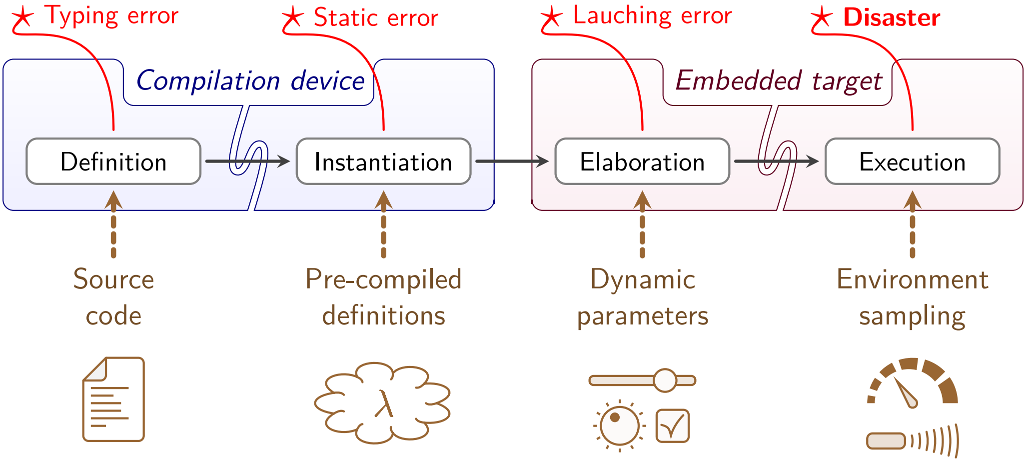
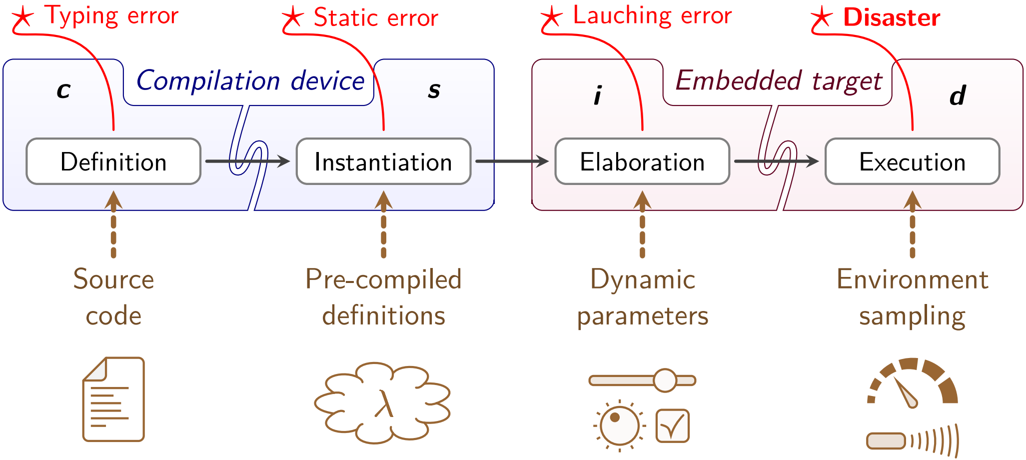


Embedded Synchronous Program Timeline



Binding-time analysis [NN88]

Conclusion

Polynomial size polymorphism

- Expressiveness / formal handling trade-off
- Decidable type and size checking
- Size reconstruction heuristics

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- Marking of remaining checks
- Multiple analyses / code generation perspectives

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Coercions

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Thank you!

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Complete syntax

η	$::=$		*Sizes*
	$\mid$	ι	variable
	$\mid$	n	constant
	$\mid$	$\eta \texttt{ + } \eta$	sum
	$\mid$	$\eta \texttt{ * } \eta$	product
τ	$::=$		*Types*
	$\mid$	α	variable
	$\mid$	$\texttt{<}\eta\texttt{>}$	singleton
	$\mid$	$\texttt{[}\eta\texttt{]}$	interval
	$\mid$	$\texttt{int}$	integer
	$\mid$	$\texttt{bool}$	boolean
	$\mid$	$\tau \to \tau$	function
σ	$::=$		*Type scheme*
	$\mid$	τ	simple type
	$\mid$	$\forall\iota.\ \sigma$	size quantif.
	$\mid$	$\forall\alpha.\ \sigma$	type quantif.

e	$::=$		*Expressions*
	$\mid$	x	variable
	$\mid$	$e\ e$	application
	$\mid$	$\lambda x{:}\tau.\ e$	abstraction
	$\mid$	$\texttt{true} \mid \texttt{false}$	boolean
	$\mid$	n	integer
	$\mid$	o	operateur
	$\mid$	$e\ {}_{\eta}$	size application
	$\mid$	$e\ {}_{\tau}$	type application
	$\mid$	$\Lambda\iota.\ e$	size abstraction
	$\mid$	$\Lambda\alpha.\ e$	type abstraction
	$\mid$	$\texttt{fix}\ x{:}\sigma = e$	fix-point
	$\mid$	$\texttt{let}\ x{:}\sigma = e\ \texttt{in}\ e$	local definition
	$\mid$	$\texttt{let size}\ \iota = e\ \texttt{in}\ e$	size definition
	$\mid$	$\texttt{<}\eta\texttt{>}$	size
	$\mid$	$e \triangleright \tau$	coercion
	$\mid$	$\texttt{case}\ e\ \texttt{then}\ e\ \texttt{else}\ e$	by case def.
	$\mid$	$.$	dead branch